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FAST TRACK COMMUNICATION

Soliton solutions of the KP equation with V-shape initial waves

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Abstract
We consider the initial value problems of the Kadomtsev–Petviashvili (KP) equation for symmetric V-shape initial waves consisting of two semi-infinite line solitons with the same amplitude. Those are particularly important for studies of large amplitude waves such as tsunami in shallow water. Numerical simulations show that the solutions of the initial value problem approach asymptotically to certain exact solutions of the KP equation found recently in [1]. We then use a chord diagram to explain the asymptotic result. This provides an analytical method to study asymptotic behavior for the initial value problem of the KP equation. We also demonstrate a real experiment of shallow water waves which may represent the solution discussed in this communication.

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1. Introduction

We consider the Kadomtsev–Petviashvili (KP) equation in the form,

$$(4u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0, \tag{1.1}$$

which describes quasi-two-dimensional, weakly nonlinear and long waves such as shallow water waves and ion acoustic waves [2–4]. It is well known that the KP equation admits N-soliton solutions, each of which has asymptotically the same set of N line solitons in both $y \to \pm\infty$. Interaction property of two line solitons with the same amplitude has been studied based on a two-soliton solution which forms an 'X' shape with a phase shift due to the interaction (see figure 1). We call this two-soliton solution O-type, where 'O' stands for original. However, in 1977, Miles [5] pointed out that O-type soliton solution becomes singular if the angle of the interaction is smaller than certain critical value. Since the KP equation is supposed to give better approximations in this regime of small angles, it sounds

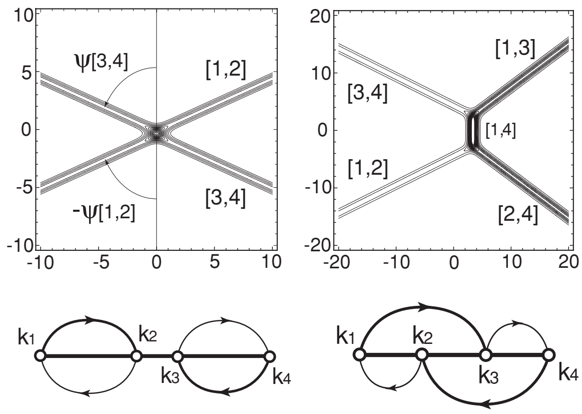

Figure 1. Contour plots and the chord diagrams of soliton solutions. Left: O-type solution. Right: (3142)-type solution. Each $[i, j]$ denotes the $[i, j]$-soliton. The length of $[1, 4]$-soliton changes in t. The upper (lower) chords represent the asymptotic solitons in $y \gg 0$ ($y \ll 0$). The thicker chords correspond to the solitons in the right side ($x \gg 0$), and the arrows on the chords show the pairings in the permutations.

strange that we do not have a reasonable solution of the equation. Miles also found that at the critical angle the two line solitons of the O-type solution resonantly interact, and a third wave (soliton) is created to make a 'Y'-shape solution, which is indeed an exact solution of the KP equation. After the discovery of the resonant phenomena in the KP equation, several numerical and experimental studies were performed (see e.g. [6–9]). However, no significant progress has been made in the study of the soliton solutions of the KP equation for almost quarter century.

In the last five years, a large variety of new soliton solutions has been found and classified [1, 10–13]. Those new soliton solutions enable us to describe the interaction properties of line solitons even in the parameter regime of small angles where O-type solution becomes singular. In this communication, we report how some of these new solutions appear under certain physical settings considered in the studies on the generation of freak (or rogue) waves in shallow water [14–16]. In particular, we show that the asymptotic solutions can be predicted by *chord diagrams*, which parametrize the exact soliton solutions [1, 13]. We also present an elementary experiment of shallow water wave demonstrating a real existence of those new solutions.

2. Soliton solutions

Let us first recall that $u(x, y, t) = 2[\ln \tau(x, y, t)]_{xx}$ is a solution to (1.1), if the τ-function, $\tau(x, y, t)$, is given by the Wronskian determinant form with respect to the x-variable,

$$\tau(x, y, t) = \mathrm{Wr}(f_1, \ldots, f_N), \tag{2.1}$$

where the functions f_i's satisfy the *linear* equations, $f_y = f_{xx}$, $f_t = -f_{xxx}$ (see e.g. [17]). The soliton solutions we consider are generated by

$$f_i = \sum_{j=1}^{M} a_{ij} e^{\theta_j}, \qquad \theta_j = k_j x + k_j^2 y - k_j^3 t. \tag{2.2}$$

Here the coefficient matrix $A = (a_{ij})$ is a constant $N \times M$ matrix and k_j are constants with the ordering $k_1 < \cdots < k_M$. Thus each solution $u(x, y, t)$ is completely determined by the A-matrix and the k-parameters.

One-soliton solution is then obtained in the case with $N = 1$ and $M = 2$: we have $\tau = e^{\theta_i} + e^{\theta_j}$ for $k_i < k_j$, and $u = 2(\ln \tau)_{xx}$ gives

$$u = \tfrac{1}{2}(k_j - k_i)^2 \operatorname{sech}^2 \tfrac{1}{2}(\theta_j - \theta_i). \tag{2.3}$$

The line $\theta_i = \theta_j$ represents the ridge of the soliton, and we refer to this *line* soliton as $[i, j]$-soliton with $i < j$ (i.e. $k_i < k_j$). The amplitude $\alpha[i, j]$ and the inclination $\gamma[i, j] := \tan \psi[i, j]$ of this soliton are given by

$$\alpha[i, j] = \tfrac{1}{2}(k_j - k_i)^2, \qquad \gamma[i, j] = k_i + k_j, \tag{2.4}$$

where $-\pi/2 < \psi[i, j] < \pi/2$ is the angle of the line soliton measured counterclockwise from the y-axis. This angle also represents the propagation direction of the line soliton (see figure 1). The wave vector $\mathbf{K}[i, j] := (K_x, K_y)$ and the frequency $\Omega[i, j]$ of the $[i, j]$-soliton in the form $u(x, y, t) = \phi(K_x x + K_y y - \Omega t)$ are given by

$$\mathbf{K}[i, j] = \left(k_j - k_i, k_j^2 - k_i^2\right), \qquad \Omega[i, j] = k_j^3 - k_i^3, \tag{2.5}$$

and they satisfy the dispersion relation $4K_x\Omega = 3K_y^2 + K_x^4$. Note also that each soliton propagates in the positive x-direction, i.e. $\Omega[i, j]/K_x[i, j] > 0$.

In the case with $N = 2$, $M = 4$, using the Binet–Cauchy theorem for the determinant, the τ-function $\tau = \mathrm{Wr}(f_1, f_2)$ can be expressed in the form

$$\tau = \sum_{1 \leqslant i < j \leqslant 4} \xi(i, j) E(i, j), \tag{2.6}$$

where $\xi(i, j)$ is the 2×2 minor consisting of ith and jth columns of the matrix A, and $E(i, j) = \mathrm{Wr}(e^{\theta_i}, e^{\theta_j}) = (k_j - k_i)e^{\theta_i + \theta_j}$. For the regular solutions, we require all of these minors to be non-negative (note $E(i, j) > 0$).

As was shown in [1, 12, 13], each τ-function (2.6) generates a soliton solution which consists of at most two line solitons for both $y \to \pm\infty$. In the cases where the τ-function (2.6) has at least four terms (i.e. A is *irreducible*, see [1, 12]), we have seven different types of soliton solutions. Two of them are usual two-soliton solutions, O-type and P-type (this type fits better in the *physical* setting for the KP equation), and they are steady translational solutions with an 'X'-shape. The other five are non-stationary solutions.

In particular, we consider the following two types which are relevant to the solutions of the initial value problems considered in this communication: one is the O-type solution which consists of two line solitons of [1, 2] and [3, 4] for $y \to \pm\infty$. Other one is non-stationary, and it consists of [1, 3] and [3, 4] line solitons for $y \to +\infty$ and [1, 2] and [2, 4] line solitons for $y \to -\infty$. Let us call this soliton (3142)-type, because those four line solitons represent a permutation $\pi = \binom{1\ 2\ 3\ 4}{3\ 1\ 4\ 2}$. It was shown in general [1, 13] that each exact solution generated by the Wronskian determinant considered in this communication can be parametrized by a unique element of the permutation group. In this representation, O-type is expressed as (2143)-type solution. Figure 1 illustrates the contour plots of O-type and (3142)-type solutions in the xy-plane, and the corresponding *chord diagrams* which represent each soliton as a chord joining a pair of k_i's following its permutation representation. The upper chords represent the asymptotic solitons $[i, j]$ for $y \gg 0$ with $j = \pi(i) > i$ and the lower chords for the asymptotic solitons $[i, j]$ for $y \ll 0$ with $i = \pi(j) < j$. Note also that the length and the location of each chord give the amplitude and the angle of the corresponding soliton (cf (2.4)). The A-matrices for those solutions are respectively given by

$$A_{\mathrm{O}} = \begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 0 & 1 & b \end{pmatrix}, \qquad A_{(3142)} = \begin{pmatrix} 1 & a & 0 & -c \\ 0 & 0 & 1 & b \end{pmatrix}, \tag{2.7}$$

where $a, b, c > 0$ are constants determining the locations of the solitons (see [13]). Note that the τ-function for the (3142)-type contains five exponential terms,

$$\tau = (k_3 - k_1)e^{\theta_1+\theta_3} + (k_4 - k_1)be^{\theta_1+\theta_4} + (k_3 - k_2)ae^{\theta_2+\theta_3} \\ + (k_4 - k_2)abe^{\theta_2+\theta_4} + (k_4 - k_3)ce^{\theta_3+\theta_4}, \tag{2.8}$$

and the τ-function for O-type with $c = 0$ in (2.8) contains only four terms.

Let us fix the amplitudes and the angles of the solitons in the positive x regions for both O- and (3142)-types, so that those solutions are symmetric with respect to the x-axis (see figure 1):

$$\alpha \equiv \begin{cases} \alpha[1,2] = \alpha[3,4] & \text{(for O-type)} \\ \alpha[1,3] = \alpha[2,4] & \text{(for (3142)-type)} \end{cases} \\ \gamma \equiv \begin{cases} -\gamma[1,2] = \gamma[3,4] > 0 & \text{(for O-type)} \\ -\gamma[1,3] = \gamma[2,4] > 0 & \text{(for (3142)-type)}. \end{cases} \tag{2.9}$$

Then from (2.4), one can find the k-parameters in terms of α and γ with $k_1 = -k_4$ and $k_2 = -k_3$ (due to the symmetry): in the case of O-type, we have

$$k_1 = -\gamma/2 - \sqrt{\alpha/2}, \qquad k_2 = -\gamma/2 + \sqrt{\alpha/2}. \tag{2.10}$$

The ordering $k_2 < k_3$ then implies $\gamma > \sqrt{2\alpha}$.

On the other hand, for the (3142)-type, we have

$$k_1 = -\gamma/2 - \sqrt{\alpha/2}, \qquad k_2 = \gamma/2 - \sqrt{\alpha/2}. \tag{2.11}$$

The ordering $k_2 < k_3$ implies $\gamma < \sqrt{2\alpha}$.

Thus, if all the solitons in the positive x -region have the same amplitude α for both O- and (3142)-types, then an O-type solution arises when $\gamma > \sqrt{2\alpha}$, and a (3142)-type when $\gamma < \sqrt{2\alpha}$. Then the limiting value at $k_2 = k_3$ $(= 0)$ defines the critical angle, i.e.

$$\gamma_c := \sqrt{2\alpha}. \tag{2.12}$$

Note from (2.8) that at the critical angle, i.e. $k_2 = k_3$, the τ-function has only three exponential terms, and this gives a 'Y'-shape resonant solution as Miles noted [5].

One can also see from (2.4) that for (3142)-type solution, the solitons in the negative x-region are smaller than those in the positive region, i.e. $\alpha[3,4] = \alpha[1,2] = \gamma^2/2 < \alpha$, and the angles of those in the negative x-regions do not depend on γ and $\gamma[3,4] = -\gamma[1,2] = \gamma_c$. Two sets of three solitons $\{[1,3],[1,4],[3,4]\}$ and $\{[2,4],[1,4],[1,2]\}$ are both in the soliton resonant state, that is, they are resonant triplets satisfying the resonant conditions, e.g.

$$\begin{cases} \mathbf{K}[1,3] + \mathbf{K}[3,4] = \mathbf{K}[1,4], \\ \Omega[1,3] + \Omega[3,4] = \Omega[1,4], \end{cases} \tag{2.13}$$

which can be easily checked from $\mathbf{K}[i,j]$ and $\Omega[i,j]$ given in (2.5). These properties of the (3142)-type solution are the same as those of Miles' asymptotic solution for the Mach reflection of a shallow water soliton [5].

3. Numerical simulation

We now discuss the numerical simulations of the KP equation with V-shape initial waves with $\alpha = 2$ and one free parameter γ $(0 < \gamma < \pi/2)$,

$$u(x, y, 0) = 2\,\mathrm{sech}^2(x - \gamma|y|). \tag{3.1}$$

This initial wave form has been considered in the study on an oblique reflection of a solitary wave in two layer fluid due to a rigid wall along the x-axis [16, 18]. The numerical simulations

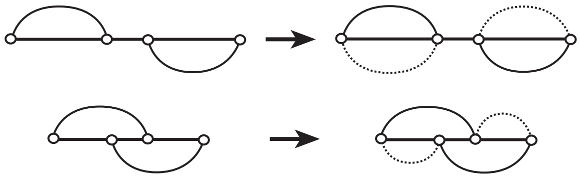

Figure 2. Minimal completion of the chord diagrams. The left diagrams show the initial V-shape waves with the same amplitude for O-type (top) and (3142)-type (bottom). The right diagrams represent the asymptotic solutions.

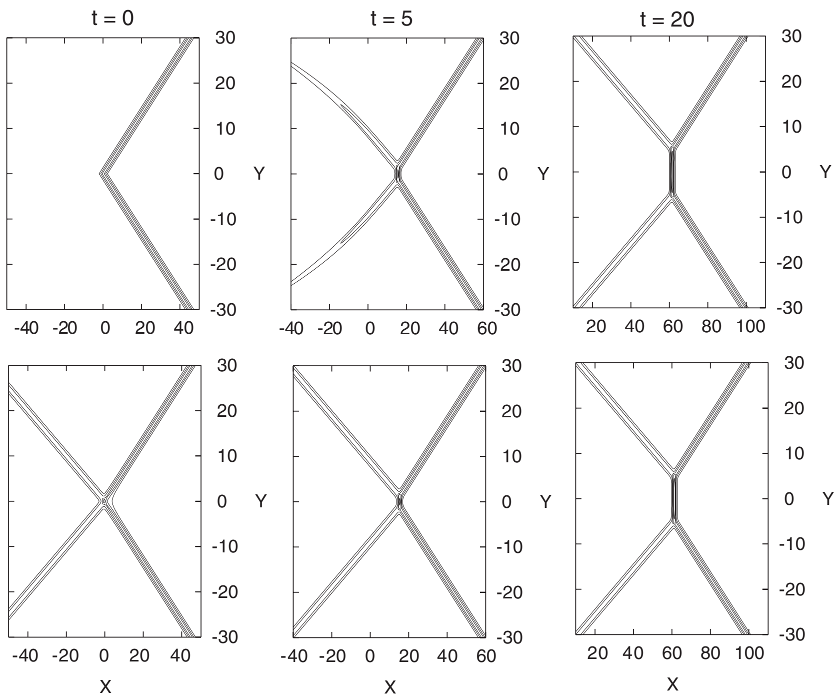

Figure 3. The contour plots of the solutions. Upper: numerical solution for V-shape initial wave with $\alpha = 2$ and $\gamma = 1.5 < \gamma_c = 2$. Lower: (3142)-type solution with the same α and γ which give $(k_1, \ldots, k_4) = (-1.75, -0.25, 0.25, 1.75)$. The parameters in the A-matrix are chosen by $(a, b, c) = (4, \frac{4}{7}, \frac{4}{3})$, so that all solitons meet at the origin at $t = 0$ (see [13]). The stem amplitude is given by $\alpha[1, 4] = \frac{1}{2}(k_1 - k_4)^2 = 6.125$.

are based on a spectral method with window technique where the boundary of the computation domain is patched by the corresponding one-soliton solutions (the details will be published elsewhere).

In terms of the chord diagrams, the V-shape initial waves are represented by the partial chords in figure 1 with the thicker chords. Then the main goal of this communication is to show that the asymptotic solutions of the simulations are given by O-type solution for $\gamma > \gamma_c$ and (3142)-type for $\gamma < \gamma_c$. Namely, the partial chords are asymptotically getting to be the corresponding complete chords (see figure 2, and the discussion in the end of this communication).

Figure 3 shows the contour plots of the numerical solutions for $\gamma = 1.5$ ($<\gamma_c = 2$) and the (3142)-type solution with the same γ. Note that the waves in the trailing part of the solution

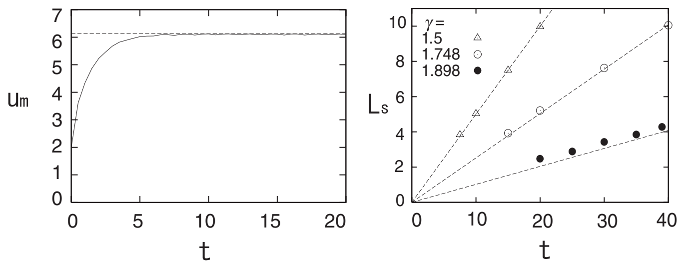

Figure 4. Left: the time evolution of the stem amplitude u_m (solid line) for $\alpha = 2, \gamma = 1.5 < \gamma_c = 2$. The dashed line denotes $\alpha[1, 4] = 6.125$. Right: the time evolution of the stem length L_s for $\alpha = 2$ and $\gamma = 1.5, 1.748, 1.898$. The dashed lines denote (3.3).

are fully developed at $t = 20$ within the sight of numerical domain. At $t = 20$ the numerical solution is in remarkable agreement with the (3142)-type solution: the waves in the trailing part in the negative x-region are almost identical with [3, 4]- and [1, 2]-solitons in $y > 0$ and $y < 0$, respectively. From the corner of the V-shape, a new wave is generated in the direction parallel to the y-axis. This wave is called *stem* and can be identified as the [1, 4]-soliton (see figure 1). The time evolution of the stem amplitude u_m is shown in figure 4, and it approaches to the asymptotic value, $\alpha[1, 4] = (\gamma + \sqrt{2\alpha})^2/2 = 6.125$. The time evolutions of the stem length L_s are also shown in figure 4. Identifying the stem as [1, 4]-soliton, the formula of L_s can be derived. The ridges of [1, 3] and [1, 4] solitons are given by $\theta_1 = \theta_3$ and $\theta_1 = \theta_4$, which lead to

$$x - \gamma y = \tfrac{1}{4}\left(\gamma_c^2 + 3\gamma^2\right)t, \qquad x = \tfrac{1}{4}(\gamma_c + \gamma)^2 t. \tag{3.2}$$

Then the stem length L_s is given by $L_s = 2y$ with y at the intersection point of those lines (3.2), i.e.

$$L_s = (\gamma_c - \gamma)t. \tag{3.3}$$

We performed simulations for the cases with $\gamma = 1.5, 1.748, 1.898 < \gamma_c = 2$, and found that the solutions agree generally well with the corresponding (3142)-type solutions. However, as seen in figure 4, the stem length L_s of the numerical solution is slightly longer than the expected value (3.3) when γ approaches to γ_c.

For $\gamma > \gamma_c = 2$, we performed the simulations for the cases with $\gamma = 2.1, 2.207, 2.367, 2.5$ and confirmed that the solutions approach to O-type solutions with the same γ. Figure 5 illustrates the case with $\gamma = 2.1$. At $t = 15$, the solution agrees quite well with the exact solution of O-type within the numerical window, that is, the waves generated in the trailing part in $x < 0$ are identified as [1, 2]- and [3, 4]-solitons with the phase shift δ_x given by [5]

$$\delta_x = \gamma_c^{-1} \ln\left[\gamma^2 / \left(\gamma^2 - \gamma_c^2\right)\right]. \tag{3.4}$$

Also using the exact solutions of O-type and (3142)-type, one can obtain the formula of the asymptotic maximum amplitude u_a in terms of γ [5],

$$u_a = \begin{cases} (\gamma + 2)^2/2 & \text{for} \quad \gamma < \gamma_c = 2, \\ 8/(1 + e^{-\delta_x}) & \text{for} \quad \gamma > \gamma_c = 2. \end{cases} \tag{3.5}$$

Figure 6 illustrates the numerical results for δ_x and u_a with formulae (3.4) and (3.5).

At the critical angle $\gamma = \gamma_c = 2$, the maximum value of u is given by $u_m = 4\alpha = 8$ (cf (3.5)). The numerical simulation shows that the amplitude approaches very slowly to the

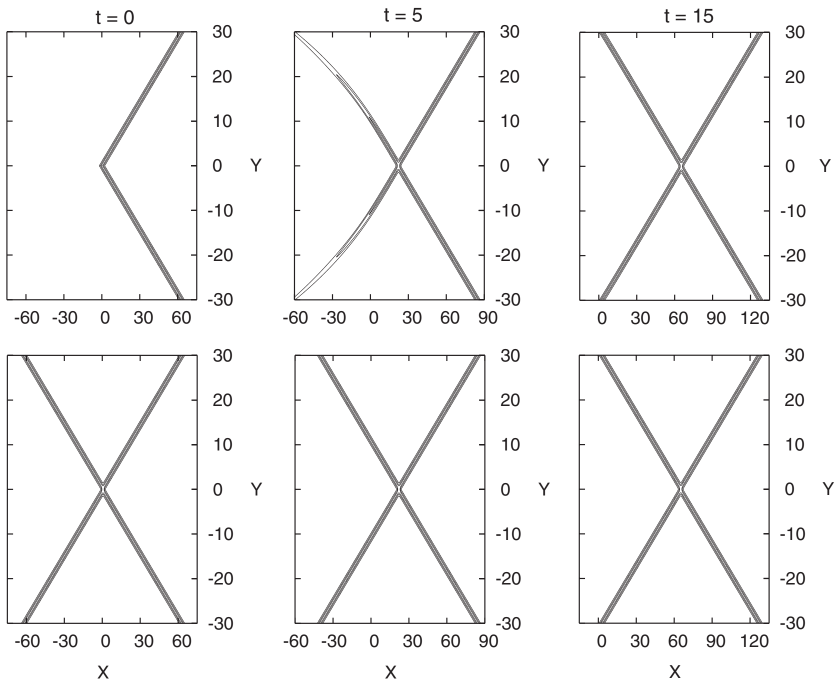

Figure 5. The contour plots of the solutions. Upper: numerical solution for V-shape initial wave with $\alpha = 2$ and $\gamma = 2.1 > \gamma_c = 2$. Lower: O-type solution with the same α and γ which gives $(k_1, \ldots, k_4) = (-2.05, -0.05, 0.05, 2.05)$. The parameters in the A-matrix are chosen by $(a, b) = (41/21, 1/21)$, so that semi-infinite line solitons in $x > 0$ meet at the origin at $t = 0$ (see [13]). The phase shift is given by $\delta_x = 1.19$, and the maximum amplitude is $u_m = 6.13$ obtained at the center of the interaction.

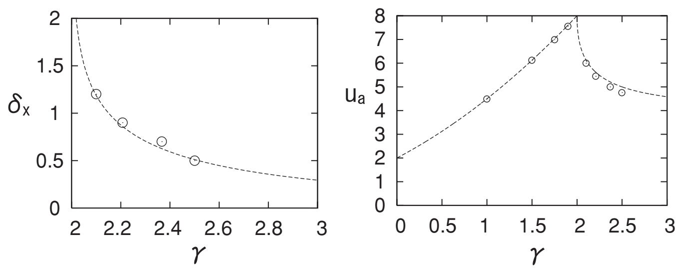

Figure 6. Left: the phase shift δ_x for O-type solution. The dashed line denotes (3.4). The circle denotes numerical results for $\gamma = 2.1, 2.207, 2.367, 2.5$. Right: the asymptotic value u_a of the maximum of $u(x, y, t)$. The dashed line denotes (3.5). The circle denotes numerical results for $\gamma = 1, 1.5, 1.748, 1.898, 2.1, 2.207, 2.367, 2.5$.

asymptotic value 8 (figure 7). In the limit $\gamma \to 2^+$ of O-type solution, the phase shifts become infinity. On the other hand, in the limit $\gamma \to 2^-$ of the (3142)-type solution, the stem length L_s approaches to zero. We could not find an exact description of the solution at the critical angle, but the simulation indicates that the phase shift (or the stem length) seems to have a logarithmic increase (figure 7).

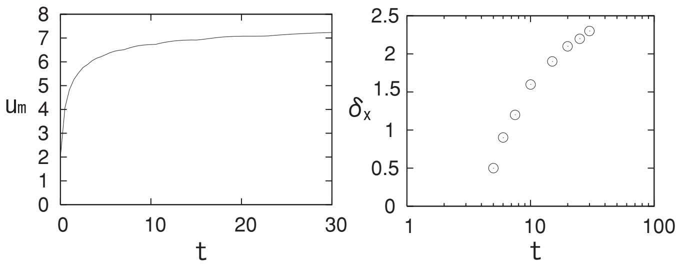

Figure 7. Left: the time evolution of the maximum u_m of $u(x, y, t)$ at the critical angle $\gamma = \gamma_c = 2$. Right: the time evolution of the phase shift δ_x at the critical angle.

Figure 8. A real experiment showing a (3142)-type solution.
(This figure is in colour only in the electronic version)

4. Summary and discussion

Let us now make a summary and some discussion: we performed several numerical simulations of the KP equation with symmetric V-shape initial waves. Then we found that the solutions are asymptotically approaching to either (3142)-type solutions or O-type solutions depending on the angles of the V-shape. Those solutions are expressed by the chord diagrams, equivalently the permutations (see figure 1). As shown in [13], the soliton solutions generated by the τ-functions in the form (2.1) are uniquely expressed by the permutations. In order to explain the asymptotic solutions, we first express each initial V-shape wave as a (sub) chord diagram consisting of two chords which correspond to the semi-infinite solitons of the V-shape. Then we observed that the asymptotic solution is given by a chord diagram which is the *completion* of the subchord diagram. Here the completion means that the resulting chord diagram should give the unique permutation and the corresponding solution has the *smallest* total length of the chords, which is referred to the *minimal* completion. For example, for the (sub) chord diagram of V-shape with $\gamma > \gamma_c$, there are *two* complete chord diagrams having this subdiagram (they are of O-type and (2413)-type). The total length of solitons in the (2413)-type is larger than the O-type one. Thus the completion of the subchord diagram is the diagram of the O-type. Although we observed numerically that the idea of 'minimal completion' can also be applied for general cases of V-shape initial waves with different amplitudes, one needs to give an analytical proof. We are currently working on this problem and will report the details in a separate communication. We would like to emphasize that one can now predict the asymptotic solution for V-shape initial wave by the method of the minimal completion. This implies that one can estimate the maximum amplitude generated by the interaction of those initial waves, and the method thus provides useful information for the study of large wave generation in systems under the assumption of long-wave and weak nonlinearity, including shallow water.

Finally, in figure 8, we present an elementary (*desktop*) experiment of a shallow water wave. The size of the tank is 37 cm × 65 cm with 2 cm depth. The waves are generated by shifting the tank in a diagonal direction. The resulting wave is similar to (3142)-type. The experiment is easy and, of course, we can also get 'Y'-shape and O-type as well.

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